

Gravitational energy from a combination of a tetrad expression and Einstein's pseudotensor

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Abstract

The energy-momentum for a gravitating system can be considered by the tetrad teleparallel gauge current in orthonormal frames. Whereas the Einstein pseudotensor used holonomic frames. Tetrad expression itself gives a better result for gravitational energy than Einstein's. Inspired by an idea of Deser, we found a gravitational energy expression which enjoys the positive energy property by combining the tetrad expression and the Einstein pseudotensor, i.e., the connection coefficient has a form appropriate to a suitable intermediate between orthonormal and holonomic frames.

1 Introduction

Gravitational energy is still a fundamental problem in general relativity. The gravitation field energy does exist through the physical phenomenon such as the Io heating, the tidal force for Jupiter acts on its satellite planet Io. Owing to the equivalence principle, it is not meaningful to study the gravitational energy at a point, however the quasilocal idea can get around this difficulty. The Bel-Robinson tensor [1] has a desirable property for a gravitating system quasilocally because it preserves the positive energy for all moving observers. Gravitational energy has been studied using the Einstein pseudotensor in holonomic frames for a long time [2]. Recently, Deser *et al.* [3] used a similar method to calculate the Landau-Lifschitz pseudotensor and obtained the Bel-Robinson tensor by making a specific combination of these two classical pseudotensors. Lately the tetrad expression [4] has been evaluated in orthonormal frames and also gave this good result. The present paper is inspired by the idea of Deser *et al.*, we found a gravitational energy expression which enjoys the positive energy property by making a specific combination of the tetrad expression and the Einstein pseudotensor, meaning that the connection coefficients are being selected by a uniquely specific intermediate between orthonormal and holonomic frames.

2 Ingredient

The curvature 2-form, in differential form, is

$$R^\alpha{}_\beta := d\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\lambda \wedge \Gamma^\lambda{}_\beta, \quad (1)$$

and the covariant derivative of the vanishing torsion is

$$0 = D\eta_\alpha{}^\beta{}_\mu = d\eta_\alpha{}^\beta{}_\mu + \Gamma^\beta{}_\lambda \wedge \eta_\alpha{}^\lambda{}_\mu - \Gamma^\lambda{}_\alpha \wedge \eta_\lambda{}^\beta{}_\mu - \Gamma^\lambda{}_\mu \wedge \eta_\alpha{}^\beta{}_\lambda, \quad (2)$$

where the dual basis $\eta^{\alpha\cdots} := *(\theta^\alpha \wedge \dots)$ and θ^α is the co-frame. At one point, it is well known that one can choose Riemann normal coordinate in which the connection coefficient satisfies

$$\Gamma^\alpha_{\beta\mu}(0) = 0, \quad -3\partial_\nu \Gamma^\alpha_{\beta\mu} = R^\alpha_{\beta\mu\nu} + R^\alpha_{\mu\beta\nu}. \quad (3)$$

Similarly, one can choose orthonormal frames [4] such that

$$\Gamma^\alpha_{\beta\mu}(0) = 0, \quad 2\partial_\nu \Gamma^\alpha_{\beta\mu} = R^\alpha_{\beta\nu\mu}. \quad (4)$$

Define the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$ and tensor $S_{\alpha\beta\mu\nu}$ in empty spacetime [2]

$$B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R_{\lambda\sigma\rho\tau} R^{\lambda\sigma\rho\tau}, \quad (5)$$

$$S_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\beta\nu}{}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu}{}^{\lambda\sigma} + \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R_{\lambda\sigma\rho\tau} R^{\lambda\sigma\rho\tau}. \quad (6)$$

3 Derive tetrad and Einstein superpotentials

Consider the first order Lagrangian (4-form) density [5]

$$\mathcal{L} := dq \wedge p - \Lambda(q, p), \quad (7)$$

where q and p are canonical conjugate variables, Λ is the potential. The corresponding Hamiltonian 3-form (density) is

$$H(N) := \mathcal{L}_N q \wedge p - i_N \mathcal{L}, \quad (8)$$

where the Lie derivative $\mathcal{L}_N := i_N d + di_N$. The interior product of the Lagrangian is

$$i_N \mathcal{L} = \mathcal{L}_N q \wedge p - \epsilon i_N q \wedge dp - \epsilon dq \wedge i_N p - i_N \Lambda - d(i_N q \wedge p), \quad (9)$$

where $\epsilon = (-1)^f$ and q is an f -form. Then (8) can be rewritten as

$$H(N) = \epsilon i_N q \wedge dp + \epsilon dq \wedge i_N p + i_N \Lambda + d\mathcal{B}(N), \quad (10)$$

where the natural boundary term is

$$\mathcal{B}(N) = i_N q \wedge p. \quad (11)$$

This is called the quasilocal boundary expression, because when one integrates the Hamiltonian density over a finite region to get the Hamiltonian, the boundary term leads to an integral over the boundary of the region. Let

$$q \rightarrow -\frac{1}{2\kappa} \eta_\alpha{}^\beta, \quad p \rightarrow \Gamma^\alpha{}_\beta. \quad (12)$$

Rewrite (11)

$$2\kappa \mathcal{B}(N) = \Gamma^\alpha{}_\beta \wedge i_N \eta_\alpha{}^\beta = -\frac{1}{2} N^\alpha U_\alpha{}^{[\mu\nu]} \epsilon_{\mu\nu}, \quad (13)$$

where N^α is the timelike vector field and the superpotential

$$U_\alpha^{[\mu\nu]} = -g^{\beta\sigma}\Gamma^\tau_{\beta\lambda}\delta^\lambda_{\tau\sigma}\delta^\mu_{\alpha\sigma}. \quad (14)$$

The connection in (14) is free for any frames as long as the superpotential gives the physical sensible results, such as inside matter (mass density) and at spatially infinity (ADM mass). It is tetrad [4] if the connection is using orthonormal frames, while Freud [6] employed holonomic frames. The corresponding gravitational energy are studied by [4] and [2] respectively. Similarly, one can use differential form to derive the boundary expression instead of using the superpotential. Furthermore, selectively combining orthonormal and holonomic frames through the connection can obtain the desirable gravitational result. This is the main issue in the present paper and the detail will be discussed in the next section.

4 Combination of tetrad and Einstein expressions

The pseudotensor can be obtained as

$$t_\alpha{}^\mu = \partial_\nu U_\alpha^{[\mu\nu]}. \quad (15)$$

Taking the second derivatives of this pseudotensor gives the gravitational energy-momentum density. Deser *et al.* [3] consider the combination of Einstein $E_{\alpha\beta}$ and Landau-Lifschitz $L_{\alpha\beta}$ pseudotensors as follows

$$\partial_{\mu\nu}^2 \left(L_{\alpha\beta} + \frac{1}{2} E_{\alpha\beta} \right) = B_{\alpha\beta\mu\nu}. \quad (16)$$

This combination gives a good result simply because the Bel-Robinson tensor. Using the differential form, consider the middle term of (13)

$$d\mathcal{B}(N) = \frac{N^\mu}{2\kappa} \left(R^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu - \Gamma^\alpha{}_\beta \wedge \Gamma^\lambda{}_\alpha \wedge \eta_\lambda{}^\beta{}_\mu - \Gamma^\alpha{}_\beta \wedge \Gamma^\lambda{}_\mu \wedge \eta_\alpha{}^\beta{}_\lambda \right). \quad (17)$$

Inspired by [3], suppose the connection with the following combination

$$\Gamma^\alpha{}_\beta = \left\{ \frac{s}{2} R^\alpha{}_{\beta\nu\mu} - \frac{k}{3} (R^\alpha{}_{\beta\mu\nu} + R^\alpha{}_{\mu\beta\nu}) \right\} x^\nu dx^\mu + \mathcal{O}(x^2), \quad (18)$$

where s, k are real numbers and $s + k = 1$ for normalization. Moreover, the curvature tensor with s refers to orthonormal frames and k means holonomic frames respectively. Using (18), rewrite (17)

$$\begin{aligned} d\mathcal{B}(N) = & -\frac{N^\mu}{2\kappa} \left\{ 2G^\rho{}_\mu + \frac{1}{4} \left[\frac{s^2 B^\rho{}_{\mu\xi\kappa} + \frac{sk}{3} (5B^\rho{}_{\mu\xi\kappa} - \frac{1}{2} S^\rho{}_{\mu\xi\kappa})}{+\frac{2k^2}{9} (4B^\rho{}_{\mu\xi\kappa} - S^\rho{}_{\mu\xi\kappa})} \right] x^\xi x^\kappa \right\} \eta_\rho \\ & + \mathcal{O}(\text{Ricci}, x) + \mathcal{O}(x^3). \end{aligned} \quad (19)$$

The first term of this expression is dominated by the Einstein tensor $G^\rho{}_\mu$ which means it satisfies the condition inside matter at the origin because of the equivalence

principle. When it goes to the higher order which indicates the gravitational energy. As mentioned before $s + k = 1$, consider the three cases. Case (i): When $k = 0$ which is the tetrad teleparallel gauge current energy-momentum expression $M_{\alpha\beta}$ for pure orthonormal frames and the associated second derivatives at the origin [4] is

$$\partial_{\xi\kappa}^2 M^\rho{}_\mu = \frac{1}{2} B^\rho{}_{\mu\xi\kappa}. \quad (20)$$

Case (ii): When $k = 1$ which is the Einstein pseudotensor and the corresponding second derivatives [2] is

$$\partial_{\xi\kappa}^2 E^\rho{}_\mu = \frac{1}{9} (4B^\rho{}_{\mu\xi\kappa} - S^\rho{}_{\mu\xi\kappa}). \quad (21)$$

Case (iii): When $k = -3$, gravitational energy-momentum density of this particular selective ratio expression $t_{\alpha\beta}$ is

$$\partial_{\xi\kappa}^2 t^\rho{}_\mu = 2B^\rho{}_{\mu\xi\kappa}. \quad (22)$$

This is a good result we found in the present paper. It only contains the pure positive Bel-Robinson tensor and the combination between orthonormal and holonomic frames are unique.

5 Conclusion

The energy-momentum for a gravitating system is considered by the tetrad teleparallel gauge current in orthonormal frames; it gives good result, namely the Bel-Robinson tensor. Likewise, the classical Einstein pseudotensor has been used in a holonomic frames to investigate the same subject, unfortunately it does not have the desired outcome. Deser *et al.* used a combination of the second derivatives of the Einstein and Landau-Lifschitz pseudotensors to obtain the Bel-Robinson tensor. Inspired by their work, we found a gravitational energy expression which enjoys the positive energy property from a combination of the tetrad expression and the Einstein pseudotensor, meaning that the connection coefficients are being selected by the uniquely specific intermediate between orthonormal and holonomic frames.

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